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A hyperscaling relation in site–bond correlated percolation

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Abstract. A novel approach is used to construct a field theoretical model of site–bond percolation. The exponents ν and η are calculated in an $\varepsilon = 6 - d$ expansion for both Ising and percolation fields, and the scaling relation $\nu_p = 2/(d-2)$ is demonstrated to hold to order one loop for $d > 4$. Using the equations of motion, the scaling relation $\nu_p = 2/(d-2+\eta_1)$ is shown to hold to all orders, extending below four dimensions. Although this scaling relation is shown to fail for $d = 1 + \varepsilon$, it is argued that it is still valid for $d \geqslant 3$.

1. Introduction

Correlated or site–bond percolation is a model of the sol–gel transition when solvent effects are present (Coniglio *et al* 1979, Tanaka *et al* 1979). In this model, clusters consist of bonds which form only between occupied sites of a lattice gas. Two types of critical behaviour (Ising and percolation) are therefore present. Below the Ising critical point ($T < T_c$), this model is in the same universality class as ordinary percolation. In this paper we are interested in critical exponents about the special point where both Ising and percolation systems are at criticality ($T = T_c$ and $p = p_c$).

Previously, this model has been studied using a Migdal approach to a Potts model (Coniglio and Klein 1980). This Potts model Hamiltonian has also been transformed into its field theoretic version and the critical exponents obtained within the ε expansion (Coniglio and Lubensky 1980).

In this paper, we treat the problem using a novel field theory, and rederive and extend the results of Coniglio and Lubensky. In particular, we derive a result for the exponent $\nu_p = \nu_1$, which describes the behaviour of the percolation correlation length ξ as $p \to p_c$ at $T = T_c$, in terms of pure Ising exponents, which we expect to be exact at $d = 3$, the physical dimensionality.

In § 2 we derive the field theory which serves as our model, then in § 3 we calculate the exponents ν and η within the ε expansion. It is also in § 3 that the relation between the exponents is obtained. Finally in § 4 we use a Migdal approach to the problem to check results in low dimensions.

2. Description of model

Site–bond percolation or correlated percolation is a model of the sol–gel transition when solvent effects are present. In ordinary bond percolation all sites are occupied and bonds are occupied at random with probability p. Clusters then consist of sites connected by occupied bonds. In site–bond percolation the sites can also be vacant

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or occupied with probabilities determined by an Ising or lattice gas Boltzmann factor (i.e. thermal averaging). Bonds can only be present between occupied nearest neighbour sites, and bond percolation is considered on the occupied lattice.

The correspondence with the sol–gel transition in the presence of solvent is made by considering vacant sites to be solvent molecules, and occupied sites to be monomers which can form chemical bonds. The percolating network of monomers is then the gel phase.

In ordinary percolation the pair correlation function $G(\boldsymbol{r}_1, \boldsymbol{r}_2)$ is the probability of percolating from $\boldsymbol{r}_1$ to $\boldsymbol{r}_2$ irrespective of the other sites in the lattice. To obtain G a simple set of diagrammatic rules suffices (Essam 1980).

(1) Form diagrams by placing oriented bonds on the lattice in such a fashion that from each site in the diagram it is possible to reach $\boldsymbol{r}_2$ by following the arrows forward, and $\boldsymbol{r}_1$ by following the arrows backward.

(2) No closed loops of arrows are allowed.

(3) Insert a factor of p for each link in the diagram.

(4) Insert a factor of (-1) for each loop in the diagram, or alternatively assign a factor of $-\mathrm{i}^n$ for each vertex where n bonds meet.

G is then the sum over all such diagrams on the lattice. Rule 2 is necessary because any diagram with a closed loop of arrows can be constructed more simply. Rule 4 is to avoid multiple countings. For example, in figure 1 the diagram (a) counts all configurations where bonds AB_1 and B_1C are present irrespective of whether AB_2 and B_2C are present. Conversely for diagram (b). The diagram (c) must then be subtracted off to avoid double counting.

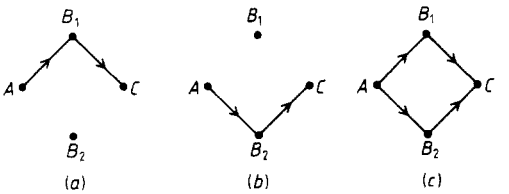

Figure 1. Diagrams contributing to $G(x, x+1)$. Diagrams (a) and (b) give contributions $+p^2$ while (c) gives a contribution $-p^4$.

In the site–bond percolation model, the sites of the lattice form a lattice gas. The occupation number of site i is n_i with possible values $n_i = 1$ (site occupied) or $n_i = 0$ (site vacant). The probability for a site to be occupied is determined by a Boltzmann factor. The set of diagrams for percolation can only be constructed on the lattice of sites which are occupied. The lattice diagrams are then thermally averaged.

A formal expression for G can be obtained if we define the commuting operators $a(\boldsymbol{r})$ and $\bar{a}(\boldsymbol{r})$ at each site $\boldsymbol{r}$ and an operation Tr such that at any site $\boldsymbol{r}$

$$\begin{aligned} &a^2 = \mathrm{i}a, \qquad \bar{a}^2 = \mathrm{i}a, \\ &\mathrm{Tr}\, a = \mathrm{Tr}\, \bar{a} = 0, \qquad \mathrm{Tr}\, a\bar{a} = n. \end{aligned} \tag{2.1}$$

Using these operators G becomes (without thermal averaging)

$$G(\boldsymbol{r}_1, \boldsymbol{r}_2) = \mathrm{Tr}\, \mathbb{P} a(\boldsymbol{r}_1) \prod_{\substack{\text{links} \\ i,j}} [1 + p\bar{a}(\boldsymbol{r}_i) a(\boldsymbol{r}_j)] \bar{a}(\boldsymbol{r}_2) \tag{2.2}$$

where $\mathbb{P}$ is an operator which destroys all diagrams with closed loops of arrows. Diagrams are created from the operators $\bar{a}(\boldsymbol{r}_i)$ and $a(\boldsymbol{r}_j)$ (where $\boldsymbol{r}_i$ and $\boldsymbol{r}_j$ are nearest neighbour links) by placing a bond on the link directed from $\boldsymbol{r}_j$ to $\boldsymbol{r}_i$. When taking the product over all such possible links, each term in the product gives rise to two sets of possible diagrams: one in which the link is present (which gives a factor p) and one in which the link is absent (which gives a factor 1). The operation Tr then eliminates diagrams which have vertices with any number of bonds going into a site and none coming out, or any number of bonds coming out of a site but none going in.

The product over links can be exponentiated as

$$\prod_{\substack{\text{links}\\ i,j}} [1+p\bar{a}(\boldsymbol{r}_i)a(\boldsymbol{r}_j)] = \prod_{\substack{\text{links}\\ i,j}} \exp(\lambda \bar{a}_i a_j) \tag{2.3}$$

where $\lambda = -\ln(1-p)$, $\bar{a}(\boldsymbol{r}_i) \equiv \bar{a}_i$ and $a(\boldsymbol{r}_j) \equiv a_j$. The exponent can be written in matrix form as

$$\prod_{\substack{\text{links}\\ i,j}} \exp(\lambda \bar{a}_i a_j) = \exp\left(\sum_{i,j} \bar{a}_i \tilde{V}_{ij} a_j\right) \tag{2.4}$$

where $\tilde{V}_{ij}$ is a matrix which accounts for the nearest neighbour interaction λ and depends on the lattice structure. We can write this exponential as a field theory via a Gaussian transformation (Amit 1978). Introducing scalar fields $\varphi(\boldsymbol{r})$ and $\bar{\varphi}(\boldsymbol{r})$ and Gaussian transforming, G becomes

$$G(\boldsymbol{r}_1, \boldsymbol{r}_2) = \mathbb{P}\ \mathrm{Tr}\ a(\boldsymbol{r}_1)\bar{a}(\boldsymbol{r}_2) \int \mathscr{D}\varphi \mathscr{D}\bar{\varphi}\ \exp\left(-\sum_{i,j} \bar{\varphi}_i \tilde{V}_{ij}^{-1} \varphi_j + \sum_i (\bar{\varphi}_i a_i + \varphi_i \bar{a}_i)\right) \tag{2.5}$$

where $\varphi_i \equiv \varphi(\boldsymbol{r}_i)$, $\bar{\varphi}_i \equiv \bar{\varphi}(\boldsymbol{r}_i)$, and as before, $\mathbb{P}$ destroys all diagrams with closed loops of arrows.

The matrix $\tilde{V}^{-1}$ is the inverse of $\tilde{V}$, the leading elements of which are

$$\tilde{V}^{-1} = z^{-1}(1 - c\nabla^2 + \ldots). \tag{2.6}$$

Here z is the coordination number of the lattice and c is a constant.

The operation Tr in (2.5) is explicitly carried out in appendix 1, § A1.1. The result is

$$G(\boldsymbol{r}_1, \boldsymbol{r}_2) = \mathbb{P} \int \mathscr{D}\varphi\, \mathscr{D}\bar{\varphi}\, n(\boldsymbol{r}_1)\varphi(\boldsymbol{r}_1)n(\boldsymbol{r}_2)\bar{\varphi}(\boldsymbol{r}_2)$$

$$\times \exp -\sum_i \{c_1 \nabla\varphi_i \cdot \nabla\bar{\varphi}_i + c_2\bar{\varphi}_i\varphi_i + n_i[\varphi_i\bar{\varphi}_i - \tfrac{1}{2}\mathrm{i}(\varphi_i\bar{\varphi}_i^2 + \bar{\varphi}_i\varphi_i^2)]\}. \tag{2.7}$$

Note that this is simply the $\langle n_1\varphi_1 n_2\bar{\varphi}_2\rangle$ correlation function with weight function $\exp[A(\varphi, \bar{\varphi})]$ where

$$A(\varphi, \bar{\varphi}) = -\sum c_1 \nabla\varphi_i \cdot \nabla\bar{\varphi}_i + c_2\bar{\varphi}_i\varphi_i + n_i[\varphi_i\bar{\varphi}_i - \tfrac{1}{2}\mathrm{i}(\varphi_i\bar{\varphi}_i^2 + \bar{\varphi}_i\varphi_i^2)]. \tag{2.8}$$

Higher-order derivatives and interaction terms in φ and $\bar{\varphi}$ higher than cubic are not shown in (2.7) and (2.8). Note that if all the n_i are set to one in (2.7) it corresponds to ordinary percolation. This becomes apparent if we consider the Feynman diagrams arising from (2.7) with all the n_i set equal to one as in figure 2. These diagrams are topologically equivalent to the diagrams for percolation constructed on the lattice. Finally, it is important to note that by including the operator $\mathbb{P}$ we ensure that the

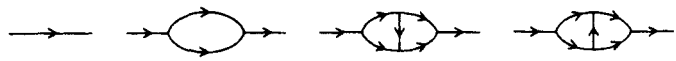

Figure 2. Lowest-order Feynman diagrams arising from (1.7) with all n_i set equal to one.

diagrammatic weights associated with each Feynman diagram are the same as those obtained in the corresponding diagrams of the $q \to 1$ limit of the Potts model (Fortuin and Kasteleyn 1969, 1972). We have explicitly checked this to the order of four loops.

To complete the model we need to thermally average over the lattice gas terms in G. To do this write the lattice gas spins (n_i) as Ising spins (s_i) related via

$$n_i = \tfrac{1}{2}(s_i + 1) \tag{2.9}$$

and average G with the Boltzmann weight (Z_1)

$$Z_1 \equiv \exp\Big(J \sum_{\substack{\text{links} \\ i,j}} s_i s_j\Big). \tag{2.10}$$

This term can be written in terms of a matrix $\tilde{U}$ as

$$Z_1 = \exp\Big(\sum_{i,j} s_i \tilde{U}_{ij} s_j\Big). \tag{2.11}$$

Introducing a scalar field $\chi(\boldsymbol{r})$ and Gaussian transforming, Z_1 becomes

$$Z_1 = \int \mathscr{D}\chi \exp\Big(-\sum_{i,j} \tfrac{1}{2}\chi_i \tilde{U}^{-1}_{ij} \chi_j + \sum_i s_i \chi_i\Big) \tag{2.12}$$

where $\tilde{U}^{-1}$ is the inverse of $\tilde{U}$ and has leading terms

$$\tilde{U}^{-1} = z^{-1}(1 - c\nabla^2 + \dots). \tag{2.13}$$

Again, z is the coordination number of the lattice and c is some constant.

Inserting the Boltzmann weight (2.10) into G and writing the n_i in terms of s_i we obtain

$$\begin{aligned}\langle G(\boldsymbol{r}_1, \boldsymbol{r}_2)\rangle = \mathbb{P} \int \mathscr{D}\varphi\, \mathscr{D}\bar{\varphi} \mathscr{D}\chi \exp\Big(-\sum_i c_1 \nabla\varphi_i \cdot \nabla\bar{\varphi}_i + c_2 \varphi_i \bar{\varphi}_i + c_3(\nabla\chi_i)^2 + c_4(\chi_i)^2\Big) \\ \times \varphi(\boldsymbol{r}_1)\bar{\varphi}(\boldsymbol{r}_2) \operatorname*{Tr}_s (s_1 s_2 + s_1 + s_2 + 1) \\ \times \exp\Big(\tfrac{1}{2}\sum_i (s_i + 1)[\varphi_i\bar{\varphi}_i - \tfrac{1}{2}\mathrm{i}(\varphi_i\bar{\varphi}_i^2 + \bar{\varphi}_i\varphi_i^2)] + \sum_i s_i\chi_i\Big).\end{aligned} \tag{2.14}$$

The trace over s is carried out in appendix 1, § A1.2. Going over to the continuum limit the result is

$$\begin{aligned}\langle G(r_1, r_2)\rangle = \mathbb{P} \int \mathscr{D}\varphi\, \mathscr{D}\bar{\varphi}\, \mathscr{D}\chi\, \varphi(\boldsymbol{r}_1)\bar{\varphi}(\boldsymbol{r}_2) \\ \times \exp\Big(-\int \mathrm{d}^d r[c_1(\nabla\varphi \cdot \nabla\bar{\varphi}) + c_2\varphi\bar{\varphi} + c_3(\nabla\chi)^2 + c_4\chi^2]\Big) \\ \times \exp\Big(\int \mathrm{d}^d r[-\tfrac{1}{2}\mathrm{i}(\varphi\bar{\varphi}^2 + \bar{\varphi}\varphi^2) + \chi\varphi\bar{\varphi} - \tfrac{1}{2}\mathrm{i}\chi(\varphi\bar{\varphi}^2 + \varphi\bar{\varphi}^2) \\ + \tfrac{1}{4}\varphi^2\bar{\varphi}^2 - \tfrac{1}{2}\mathrm{i}(\varphi\bar{\varphi}^3 + \varphi\bar{\varphi}^3) + \chi^2\varphi\bar{\varphi}]\Big).\end{aligned} \tag{2.15}$$

This is an important result, since from this we can infer the Lagrangian for this theory

$$S(\varphi, \bar{\varphi}, \chi) = \int \mathrm{d}^d r[(\nabla\varphi \cdot \nabla\bar{\varphi}) + \tfrac{1}{2}(\nabla\chi)^2 + m_\varphi \varphi\bar{\varphi} + m_1\chi^2 + \tfrac{1}{2}\mathrm{i}g(\varphi\bar{\varphi}^2 + \bar{\varphi}\varphi^2) + u\chi\varphi\bar{\varphi} + w\chi^4 + \tfrac{1}{2}\mathrm{i}h\chi(\varphi\bar{\varphi}^2 + \bar{\varphi}\varphi^2) + k\chi^2\varphi\bar{\varphi} + m\varphi^2\bar{\varphi}^2 + \tfrac{1}{2}\mathrm{i}n(\varphi\bar{\varphi}^3 + \bar{\varphi}\varphi^3)]. \tag{2.16}$$

This also defines the coupling constants associated with each interaction term. Using this Lagrangian we can apply the renormalisation group and obtain the critical exponents ν and η for both the Ising and percolation fields.

3. Renormalisation group in ε expansion

Having obtained a Lagrangian for this theory (1.16), we can apply the renormalisation group (RG) within the ε expansion and obtain the critical exponents ν and η. A program for such calculations can be found in several sources (Amit 1978).

The upper critical dimensionality (d_c) of a theory is found by considering the canonical dimension of the lowest order coupling. From (1.16) we see that g and u are cubic couplings and so $d_c = 6$. The expansion parameter, ε, is therefore given by $\varepsilon = 6 - d$. Also by dimensional considerations, we can see that fourth- and higher-order couplings are irrelevant for $d > 4$, at the Gaussian fixed point.

Since there are two independent fields involved in the Lagrangian, there are two sets of critical exponents ν and η which can be obtained. To distinguish between these two, the subscripts 1 and 2 will be used. Subscript '1' will refer to percolation terms while subscript '2' will refer to Ising terms.

The β functions are defined as

$$\beta_g(g_0, u_0) \equiv \Lambda\, \partial g_0/\partial\Lambda|_{g_R, u_R} \tag{3.1a}$$

and

$$\beta_u(g_0, u_0) \equiv \Lambda\, \partial u_0/\partial\Lambda|_{g_R, u_R} \tag{3.1b}$$

where Λ is the cut-off which regularises the theory, the subscript '0' refers to the dimensionless bare coupling constant, and the subscript 'R' refers to the dimensionful renormalised coupling constant.

Again since there are two independent fields, the vertex functions will be written as $\Gamma^{(N,M)}$ where N implies the number of truncated percolation propagators and M implies the number of truncated Ising propagators. It is important to note that $\Gamma^{(N,M)}$ is not a composite vertex function. Composite vertex functions will be designated $\Gamma^{(N,L;M,K)}$ where L refers to the number of insertions of $\varphi\bar{\varphi}$ into percolation (φ) lines, and K refers to the number of insertions of χ^2 into Ising (χ) lines.

The renormalised vertex functions can be written as

$$\Gamma_R^{(N,M)} = Z_\varphi^{N/2} Z_\chi^{M/2} \Gamma^{(N,M)}. \tag{3.2}$$

Here Z_φ and Z_χ are the field renormalisation constants for the percolation field and the Ising field respectively. These in turn are defined from the normalisation conditions

$$(\partial/\partial(k^2))\Gamma_R^{(2,0)}|_{k=\varkappa} = 1, \qquad (\partial/\partial(k^2))\Gamma_R^{(0,2)}|_{k=\varkappa} = 1 \tag{3.3a, b}$$

where $\varkappa$ is the normalisation point. Using (3.3a) and (3.3b) we obtain respectively

$$Z_\varphi = [(\partial/\partial(k^2))\Gamma^{(2,0)}]^{-1}, \qquad Z_\chi = [(\partial/\partial(k^2))\Gamma^{(0,2)}]^{-1}. \tag{3.4a, b}$$

The diagrams contributing to $\Gamma^{(2,0)}$ to order one loop are shown in figure 3. The first correction to $\Gamma^{(0,2)}$ is of order two loops. It is important to note that the renormalisation of the Ising field contains no percolation fields, since there can be no closed loops of arrows. This occurs to all orders in perturbation theory.

Figure 3. Feynman diagrams contributing to the vertex function $\Gamma^{(2,0)}$ to order of one loop.

Doing the Feynman integrals of the diagrams to order of one loop we obtain

$$Z_\varphi = 1 + A(\Lambda)(\tfrac{1}{6}g^2 - \tfrac{1}{3}u^2), \tag{3.5a}$$

$$Z_\chi = 1, \qquad d > 4 \tag{3.5b}$$

where $A(\Lambda) = (4\pi)^{-3} \ln(\Lambda/\varkappa)$.

The renormalised coupling constants g_R and u_R are defined from the normalisation conditions

$$g_R = \Gamma_R^{(3,0)} = Z_\varphi^{3/2}\Gamma^{(3,0)}|_{SP}, \tag{3.6}$$

$$u_R = \Gamma_R^{(2,1)} = Z_\varphi Z_\chi^{1/2}\Gamma^{(2,1)}|_{SP}, \tag{3.7}$$

where the momenta k_i are chosen at a symmetry point (SP). That is

$$\boldsymbol{k}_i \cdot \boldsymbol{k}_j = \tfrac{1}{4}\varkappa^2(4\delta_{ij} - 1). \tag{3.8}$$

The diagrams contributing to $\Gamma^{(3,0)}$ and $\Gamma^{(2,1)}$ to order of one loop are shown in figure 4.

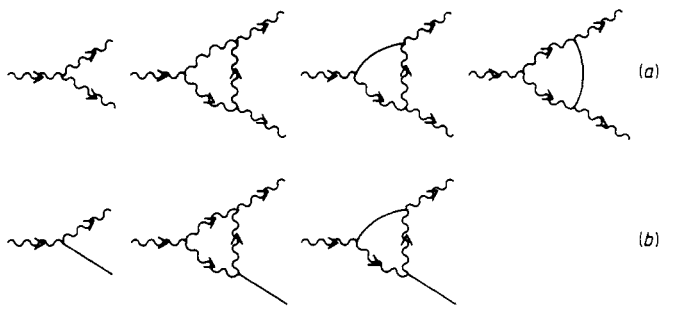

Figure 4. Feynman diagrams contributing to the vertex functions (*a*) $\Gamma^{(3,0)}$ and (*b*) $\Gamma^{(2,1)}$ to order of one loop.

Doing the Feynman integrals for $\Gamma^{(3,0)}$ and using (3.5*a*) and (3.6) we obtain

$$g_R = \Lambda^{\epsilon/2} g_0[1 + A(\Lambda)(\tfrac{5}{2}u_0^2 - \tfrac{7}{4}g_0^2)]. \tag{3.9}$$

Similarly for $\Gamma^{(2,1)}$ and using (3.5*b*) and (3.7) we obtain

$$u_R = \Lambda^{\epsilon/2} u_0[1 + A(\Lambda)(\tfrac{2}{3}u_0^2 - \tfrac{5}{6}g_0^2)]. \tag{3.10}$$

These give for the β functions

$$\beta_g(g_0, u_0) = -\tfrac{1}{2}\varepsilon g_0 + [g_0/(4\pi)^3](\tfrac{7}{4}g_0^2 - \tfrac{5}{2}u_0^2), \tag{3.11a}$$

$$\beta_u(g_0, u_0) = -\tfrac{1}{2}\varepsilon u_0 + [u_0/(4\pi)^3](\tfrac{5}{6}g_0^2 - \tfrac{2}{3}u_0^2). \tag{3.11b}$$

The fixed point occurs when $\beta_g = \beta_u = 0$. Equations (3.11) have three fixed points: (i) the percolation fixed point with $u_0 = 0$ and $g_0^2 = 2\varepsilon/7(4\pi)^3$, (ii) an unphysical fixed point with $g_0 = 0$ and $u_0^2 = -3\varepsilon/4(4\pi)^3$, and (iii) a fixed point corresponding to simultaneous gelation and phase separation (Coniglio and Lubensky 1980) with

$$g_0^2 = (4\pi)^3\varepsilon, \qquad u_0^2 = \tfrac{1}{2}(4\pi)^3\varepsilon. \tag{3.12}$$

The exponent η_i, the anomalous dimension of the field i ($i = 1, 2$), is defined as

$$\eta_i = [-\Lambda(\partial/\partial\Lambda) \ln Z_i]^* \tag{3.13}$$

where * implies that the result is taken at the fixed point. Using (3.5*a*) in (3.13) we obtain

$$\eta_1 = (4\pi)^{-3}[-\tfrac{1}{6}g_0^2 + \tfrac{1}{3}u_0^2]^* \tag{3.14}$$

which when evaluated at the fixed point gives

$$\eta_1 = 0. \tag{3.15}$$

Similarly, using (3.5*b*) in (3.13) we obtain the trivial result

$$\eta_I \equiv \eta_2 = 0, \qquad d > 4. \tag{3.16}$$

Note that (3.16) is valid to all orders in perturbation theory for all $d > 4$. This is a consequence of the fact that percolation fields do not affect Ising fields. That is, Z_χ is a power series in couplings which are pure Ising, the lowest order of which, w, is a quartic coupling. This coupling is irrelevant for $d > 4$ and hence (3.5*b*) and (3.16) are valid to all orders. Contrarily, (3.15) is not expected to be valid even to the next order in perturbation theory for $4 < d < 6$, although we have not proven this to be true.

To obtain the exponents ν_1 and ν_2, we must first obtain the renormalisation constants $Z_{\varphi\bar{\varphi}}$ and Z_{χ^2}, respectively. These are defined in terms of the bare composite vertex functions $\Gamma^{(N,L;M,K)}$. The renormalised composite vertex functions can be written as

$$\Gamma_R^{(N,L;M,K)} = (Z_\varphi)^{N/2}(Z_{\varphi\bar{\varphi}})^L(Z_\chi)^{M/2}(Z_{\chi^2})^K\Gamma^{(N,L;M,K)}. \tag{3.17}$$

Then $Z_{\varphi\bar{\varphi}}$ is found from the normalisation condition, at the scale $\varkappa$

$$\Gamma_R^{(2,1;0,0)} = 1 = Z_\varphi Z_{\varphi\bar{\varphi}}\Gamma^{(2,1;0,0)} \tag{3.18}$$

i.e.

$$Z_{\varphi\bar{\varphi}} = [Z_\varphi\Gamma^{(2,1;0,0)}]^{-1}. \tag{3.19}$$

Similarly Z_{χ^2} is found from the normalisation condition

$$\Gamma_R^{(0,0;2,1)} = 1 = Z_\chi Z_{\chi^2}\Gamma^{(0,0;2,1)} \tag{3.20}$$

i.e.

$$Z_{\chi^2} = [Z_\chi\Gamma^{(0,0;2,1)}]^{-1} = [\Gamma^{(0,0;2,1)}]^{-1}. \tag{3.21}$$

The diagrams contributing to $\Gamma^{(2,1;0,0)}$ to order of one loop are shown in figure 5.

Figure 5. Feynman diagrams contributing to the vertex function $\Gamma^{(2,1;0,0)}$ to order of one loop.

From these we find

$$Z_{\varphi\bar{\varphi}} = 1 - A(\Lambda)[\tfrac{2}{3}u_0^2 - \tfrac{5}{6}g_0^2]. \tag{3.22}$$

There are no contributions to $\Gamma^{(0,0;2,1)}$ to order of one loop since $\Gamma^{(0,2)}$ has no contributions to this order. Hence we find the trivial result

$$Z_{\chi^2} = 1, \qquad d > 4. \tag{3.23}$$

The critical exponents ν_i $(i = 1, 2)$ are defined from

$$\nu_1^{-1} = [2 - \Lambda(\partial/\partial\Lambda) \ln Z_{\varphi\bar{\varphi}}]^*, \qquad \nu_2^{-1} = [2 - \Lambda(\partial/\partial\Lambda) \ln Z_{\chi^2}]^*. \tag{3.24a,b}$$

Using (3.22) and (3.24*a*) and evaluating at the fixed point we obtain

$$\nu_1 = (2 - \varepsilon/2)^{-1} = \tfrac{1}{2}(1 + \tfrac{1}{4}\varepsilon). \tag{3.25}$$

From (3.23) and (3.24*b*) we find the trivial result

$$\nu_2 = \tfrac{1}{2} \tag{3.26}$$

which is valid to all orders in perturbation theory for $d > 4$, i.e. this is the mean field result of the Ising model. On the basis of one-loop calculations using the Potts model, Coniglio and Lubensky (1980) have conjectured the relation

$$\nu_1 = 2/(d-2), \qquad 4 \leqslant d \leqslant 6. \tag{3.27}$$

This agrees with our result, equation (2.25), also calculated to order one loop. This can be shown to be true to all orders by noting a relation between $\Gamma^{(2,1)}$ and $\Gamma^{(2,1;0,0)}$ valid to all orders, i.e.

$$\Gamma^{(2,1)} = u\Gamma^{(2,1;0,0)}. \tag{3.28}$$

Substituting this into the expression for u_R and using (3.19) we obtain

$$u_R = u(Z_{\varphi\varphi})^{-1}. \tag{3.29}$$

Here u is the dimensionful bare coupling constant whose canonical dimension is given by

$$u = u_0 \Lambda^{-d/2+3} = u_0 \Lambda^{\varepsilon/2}. \tag{3.30}$$

Taking the natural log of (3.29) and $\Lambda\,\partial/\partial\Lambda|_{u_R}$ of that result we obtain at the fixed point

$$0 = -\tfrac{1}{2}d + 3 + \nu_1^{-1} - 2 \tag{3.31}$$

or

$$\nu_1 = 2/(d-2) \tag{3.27'}$$

as before.

That relation (3.28) is true to all orders can be confirmed by considering the equations of motion. The equations of motion can be constructed from the Lagrangian by taking correlations of the form

$$\langle J(\varphi, \bar{\varphi}) \delta\mathscr{L}/\delta\chi \rangle_{\text{conn}} = 0 \tag{3.32}$$

where $J(\varphi, \bar{\varphi})$ is some function of φ and $\bar{\varphi}$, and $\delta\mathscr{L}/\delta\chi$ is the functional derivative of the Lagrangian with respect to the field χ. Only connected diagrams contributing to this correlation function are to be considered, as indicated by the subscript in (3.32).

To confirm (3.28) we use J of the form

$$J(\varphi, \varphi) = \bar{\varphi}(x)\varphi(z) \tag{3.33}$$

and from (2.16) we obtain

$$\delta\mathscr{L}/\delta\chi = -\nabla^2\chi + {m_1}^2\chi + u\varphi\bar{\varphi} + 4w\chi^3 + \tfrac{1}{2}\mathrm{i}h(\bar{\varphi}\varphi^2 + \varphi\bar{\varphi}^2) + 2k\chi\bar{\varphi}\varphi. \tag{3.34}$$

The equation of motion then gives

$$\begin{aligned}(\nabla^2 - m_1^2)\langle\bar{\varphi}(x)\chi(y)\varphi(z)\rangle &= u\langle\bar{\varphi}(x)\varphi(y)\bar{\varphi}(z)\rangle + 4w\langle\bar{\varphi}(x)\chi(y)\chi(y)\chi(y)\varphi(z)\rangle \\ &\quad + \tfrac{1}{2}\mathrm{i}h[\langle\bar{\varphi}(x)\bar{\varphi}(y)\bar{\varphi}(y)\varphi(y)\varphi(z)\rangle + \langle\bar{\varphi}(x)\varphi(y)\varphi(y)\bar{\varphi}(y)\varphi(z)\rangle] \\ &\quad + 2k\langle\bar{\varphi}(x)\chi(y)\bar{\varphi}(y)\varphi(y)\varphi(z)\rangle.\end{aligned} \tag{3.35}$$

Above $d = 4$, the quartic couplings w, h and k are all irrelevant. Therefore for $d > 4$ (3.35) reduces to

$$(\nabla^2 - m_1^2)\langle\bar{\varphi}(x)\chi(y)\varphi(z)\rangle = u\langle\varphi(x)\varphi(y)\bar{\varphi}(y)\varphi(z)\rangle. \tag{3.36}$$

Since the left-hand side of (3.36) gives rise to the diagrams contributing to $\Gamma^{(2,1)}$ and the right-hand side gives rise to the diagrams contributing to $\Gamma^{(2,1;0,0)}$, equation (3.36) confirms (3.28) to all orders.

Coniglio and Lubensky (1980) speculate that the relation $\nu_1 = 2/(d-2)$ breaks down below $d = 4$ due to a breakdown in the classical scaling of one of the fields. In this theory that corresponds to the quartic couplings becoming relevant. Within a one-loop calculation about $d = 6$, the couplings h and k are irrelevant even below $d = 4$ (see appendix 2). Therefore for $d \leqslant 4$ we shall consider only the first two terms on the right-hand side of (3.35). That is

$$(\nabla^2 - m_1^2)\langle\bar{\varphi}(x)\chi(y)\varphi(z)\rangle = u\langle\bar{\varphi}(x)\varphi(y)\bar{\varphi}(y)\varphi(z)\rangle + 4w\langle\bar{\varphi}(x)\chi(y)\chi(y)\chi(y)\varphi(z)\rangle. \tag{3.37}$$

A scaling relation can be obtained by considering how each correlation function in (3.37) scales. This in turn can be accomplished by considering the partial differential equation which arises from the statement of renormalisability of the theory. That is the renormalised vertex function approaches a finite limit as $\Lambda \to \infty$ and therefore to leading order is independent of Λ. Hence

$$\Lambda\,\partial/\partial\Lambda|_{g_R, u_R}\Gamma_R^{(N,M)} = 0. \tag{3.38}$$

Combining (3.2) with (3.38) we obtain for the vertex function implied on the left-hand side of (3.37)

$$(\Lambda\,\partial/\partial\Lambda - \eta_1 - \tfrac{1}{2}\eta_2)\Gamma^{(2,1)} = 0 \tag{3.39}$$

which has solution

$$\Gamma^{(2,1)} = \Lambda^{\eta_1 + \eta_2/2}\Phi_1(k_i). \tag{3.40}$$

Similarly, combining (3.17) with (3.38) we obtain for the vertex function implied by the first term on the right of (3.37)

$$(\Lambda\,\partial/\partial\Lambda - \eta_1 + 2 - \nu_1^{-1})\Gamma^{(2,1;0,0)} = 0 \tag{3.41}$$

which has solution

$$\Gamma^{(2,1;0,0)} = \Lambda^{\eta_1 - (2 - \nu_1^{-1})}\Phi_2(k_i). \tag{3.42}$$

Finally, for the second vertex function implied on the right of (3.37) we obtain

$$[\Lambda\,\partial/\partial\Lambda - \eta_1 - \tfrac{1}{2}\eta_2 + (\varepsilon - \tfrac{1}{2}\eta_2)]\Gamma_{\chi^3}^{(2,1)} = 0 \tag{3.43}$$

which has solution

$$\Gamma_{\chi^3}^{(2,1)} = \Lambda^{\eta_1+\eta_2/2-(\varepsilon-\eta_2/2)}\Phi_3(k_i). \tag{3.44}$$

The term $(\varepsilon - \eta_2/2)$ in (3.43) arises from considering the χ^3 insertion in this vertex function. It is calculated by considering the equation of motion arising from the correlation

$$\langle \chi^3(y)\delta\mathscr{L}/\delta\chi\rangle_{\text{conn}} = 0. \tag{3.45}$$

Using (3.40), (3.42) and (3.44) in (3.37) and taking the large Λ limit gives the scaling relation

$$\eta_1 + \tfrac{1}{2}\eta_2 = \tfrac{1}{2}\varepsilon + \eta_1 - (2 - \nu_1^{-1}) \tag{3.46}$$

or

$$\nu_1 = 2/(d - 2 + \eta_2). \tag{3.47}$$

Since $\eta_2 = 0$ for $d > 4$, this reduces to $\nu_1 = 2/(d-2)$ for $d > 4$ in agreement with (3.27). Thus (3.47) should be valid for all d and to all orders, if the assumption that the quartic couplings h and k are indeed irrelevant below $d = 4$ is correct. As we will show in § 3, a breakdown of (3.47) does occur somewhere below $d = 4$ as this scaling relation does not hold near $d = 1$.

4. Real space renormalisation group near $d = 1$

To test the scaling relation $\nu_1 = 2/(d - 2 + \eta_2)$ for $d < 4$ we consider a d-dimensional lattice, perform a Migdal–Kadanoff real space renormalisation group (RSRG) calculation (Migdal 1976, Kadanoff 1976), and evaluate at $d = 1 + \varepsilon$. To leading order in ε, we expect this approximation to yield correct results. To obtain the renormalisation group (RG) equations we consider a lattice where nearest neighbour sites are coupled by an Ising (lattice gas) parameter J, and bonds between near neighbour sites are present with probability p.

In ordinary percolation one considers cells in the lattice consisting of b bonds on an edge. Within each cell there are a total of db^d bonds. To perform a Migdal–Kadanoff RSRG transformation the bonds within the cell are 'moved' to the edges to form a super cell; that is, the edges of each super cell consist of b groups of bonds in series, each group containing b^{d-1} bonds in parallel as in figure 6. The RG equation for p can be obtained by dedecoration of the b^d bonds along an edge. Since percolation

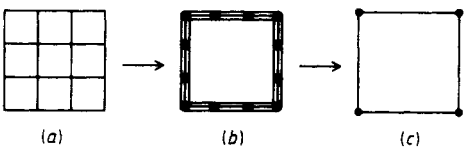

Figure 6. Bond moving in the Migdal–Kadanoff RSRG scheme. $(a) \rightarrow (b)$ bond moving for $d = 2$, $b = 3$. $(b) \rightarrow (c)$ decimation.

across any one of the b^{d-1} bonds in parallel can occur with probability p, the (rescaled) probability that percolation can occur across the parallel group is (Kirkpatrick 1977)

$$p'_{\text{par}} = 1-(1-p)^{b^{d-1}}. \tag{4.1}$$

The (rescaled) probability that percolation can occur across b bonds in series if the probability for percolation across a single bond is p is (Kirkpatrick 1977)

$$p'_{\text{ser}} = p^b. \tag{4.2}$$

Combining (4.1) and (4.2) we find that the RG equation for p in ordinary percolation is

$$p' = [1-(1-p)^{b^{d-1}}]^b. \tag{4.3}$$

In site–bond percolation the probability for percolation is not simply p. Given that the spin at one site is up (or that that site is occupied in the lattice gas model) then the probability for percolation to a neighbouring site is

$$p_{\text{sb}} = p[e^J/(e^J+e^{-J})]. \tag{4.4}$$

The second term on the right-hand side of (4.4) represents the probability that the neighbouring site also has spin up. Because of the nature of the bond moving scheme, we cannot simply substitute (4.4) for p in (4.3). Rescaling the coupling J for b^{d-1} bonds in parallel we obtain

$$J'_{\text{par}} = b^{d-1}J. \tag{4.5}$$

The probability then for percolating across a group of b^{d-1} bonds in parallel given that one spin is up is

$$p'_{\text{par}} = [1-(1-p)^{b^{d-1}}]\left(\frac{\exp(b^{d-1}J)}{\exp(b^{d-1}J)+\exp(-b^{d-1}J)}\right). \tag{4.6}$$

Since there are b of these parallel groups in series the RG equation for p becomes

$$p'\frac{e^{J'}}{e^{J'}+e^{-J'}} = [1-(1-p)^{b^{d-1}}]^b\left(\frac{\exp(b^{d-1}J)}{\exp(b^{d-1}J)+\exp(-b^{d-1}J)}\right)^b. \tag{4.7}$$

Defining $r \equiv b^{d-1}$ and rearranging, this becomes

$$p' = (1+e^{-2J'})(1+e^{-2rJ})^{-b}[1-(1-p)^r]^b. \tag{4.8}$$

The rescaling of J in the Ising model has been studied by Kadanoff (1976) and is given by

$$J' = \tanh^{-1}(\tanh rJ)^b. \tag{4.9}$$

The unstable fixed point of (4.9) to leading order in ε is

$$J^* \simeq (2\varepsilon)^{-1} - \tfrac{1}{4}\ln b = -\tfrac{1}{2}\ln(b^{1/2}e^{-1/\varepsilon}). \tag{4.10}$$

Using this in the terms containing J in (4.8) we find to leading order in ε

$$(1+e^{-2J^*})(1+e^{-2rJ^*})^{-b} \simeq 1-be^{-2/\varepsilon}. \tag{4.11}$$

Therefore the unstable fixed point of (4.8) to leading order in ε is

$$p^* \simeq 1-b^{1/2}e^{-1/\varepsilon} \tag{4.12}$$

which is the same as in pure percolation. Note that to this order, the fixed points given in (4.10) and (4.12) are related as $p^* = 1-e^{-2J^*}$. This agrees with the conjecture

made by Coniglio and Klein (1980) on the basis of a Potts model approach to this problem. The leading exponential behaviour of p^*, J^* is independent of b, indicating the reliability of this term.

Linearising (3.8) about the fixed point gives

$$\lambda_1 = (1+e^{-2J^*})(1+e^{-2rJ^*})^{-b} b^{\varepsilon+1}(1-p^*)^{r-1}[1-(1-p^*)^r]^{b-1}. \tag{4.13}$$

Evaluating (4.13) at the fixed point (4.10) and (4.12), we find to leading order the eigenvalue $y_1 \simeq \varepsilon$. Thus

$$\nu_1 = y_1^{-1} \simeq 1/\varepsilon. \tag{4.14}$$

We are now interested in obtaining the exponent η_2. A simple way to get η_2 is to assume that the RG equation for the magnetic field H is an analytic function of H. That is

$$H' = f(J, d, b)H + O(H^2) \tag{4.15}$$

where $f(J, d, b)$ is some function of J, the dimension d and the rescaling factor b. Since we are interested only in the lowest-order term in ε, we expand f about $d = 1$:

$$f(J, b, d) \simeq f(J, b, 1) + \partial f(\infty, b, d)/\partial d|_{d=1}\varepsilon. \tag{4.16}$$

The derivative with respect to d is taken of the function f which is evaluated at $J \to \infty$, again to retain only lowest-order terms.

The functions $f(J, b, 1)$ and $f(\infty, b, d)$ can be evaluated by exact means. By the Nauenberg–Nienhuis (1974) criterion, at $T = 0$, i.e. $J \to \infty$, the magnetic eigenvalue of the Ising model is $y_H = d$. Thus $\lambda_H = b^d$ and therefore $H' = b^d H$. This gives $f(\infty, b, d)$ directly as

$$f(\infty, b, d) = b^d. \tag{4.17}$$

Similarly, in one dimension H rescales under a decimation transformation (using $b = 2$) as

$$H' = H(1+\tanh 2J). \tag{4.18}$$

Therefore

$$f(J, 2, 1) = (1+\tanh 2J). \tag{4.19}$$

Using (4.17) and (4.19) we get for $f(J, b = 2, d)$

$$f(J, 2, d) \simeq (1+\tanh 2J) + \varepsilon 2 \ln 2. \tag{4.20}$$

Evaluating (4.20) at the fixed point (4.10) we find

$$f(J, 2, \varepsilon) \simeq 2 - 4e^{-2/\varepsilon} + \varepsilon 2 \ln 2 \tag{4.21}$$

which gives the Ising magnetic eigenvalue

$$y_2^H \simeq 1 + \varepsilon - (2/\ln 2)\, e^{-2/\varepsilon}. \tag{4.22}$$

We have used $b = 2$ in obtaining y_2^H, (4.22), since we used $b = 2$ in obtaining $f(J, b, 1)$, (4.19), and since there is no strong b dependence in the fixed point (4.10). That is, b appears only as a coefficient and not within the exponent of J^*. Note that this result, (4.22), agrees with the result obtained by Bruce and Wallace (1981) using a droplet model approach to an Ising system in $d = 1 + \varepsilon$ dimensions.

Now η_2 is related to the magnetic eigenvalue y_2^H via

$$\eta_2 = d + 2 - 2y_2^H \tag{4.23}$$

so that

$$d - 2 + \eta_2 = 2(d - y_2^H). \tag{4.24}$$

Using the expression for y_2^H, (4.22), in (4.24) and noting that $d = 1 + \varepsilon$, we find

$$d - 2 + \eta_2 \simeq (4/\ln 2)\, \mathrm{e}^{-2/\varepsilon}. \tag{4.25}$$

Comparing with ν_1, (4.14), we see that the scaling expression $\nu_1 = 2/(d - 2 + \eta_2)$ does not hold near $d = 1$.

Coniglio and Klein (1980) have studied site–bond correlated percolation using a Migdal–Kadanoff decimation of a Potts model in two dimensions. Based on their results for ν_1 ($= 0.535$), and the exact result for the Ising model in two dimensions $\eta_2 = \frac{1}{4}$, the scaling relation $\nu_1 = 2/(d - 2 + \eta_2)$ apparently does not hold for $d = 2$. This discrepancy cannot be accounted for by appealing to the uncertainty which arises naturally when the Migdal–Kadanoff approximation is used.

It is possible that this failure of the scaling relation in $1 + \varepsilon$ and two dimensions can be explained in part by the work of Fucito and Parisi (1981). They showed that the fixed point of a pure percolation model in two dimensions is unstable with respect to a φ^4 interaction. That is, the coupling associated with this interaction becomes relevant in two dimensions. If this is also true of the quartic couplings in our model, then it may be reasonable to assume that the scaling relation is still valid in three dimensions.

Finally, we would like to find a value for η_1 near $d = 1$ to compare with that obtained to order one loop near $d = 6$, namely $\eta_1 = 0$. To do this, we use the same procedure as used in obtaining η_2. That is, assume that the RG equation for the ghost field H is an analytic function of H:

$$H' = f(p, J, d, b)H + \mathrm{O}(H^2). \tag{4.26}$$

Again, expand f about $d = 1$ and get

$$f(p, J, b, d) \simeq f(p, J, b, 1) + \partial f(p = 1, J = \infty, b, d)/\partial d|_{d=1}\varepsilon. \tag{4.27}$$

As before, $f(p = 1, J = \infty, b, d) = b^d$ and we are left to evaluate $f(p, J, b, 1)$.

The function $f(p, J, b, 1)$ is obtained by considering paths for percolation to the ghost site from a given site on the lattice. If we take only first-order terms in H, since we are working near the fixed point $H = 0$, then the probability P of percolating from a given site to the ghost site is

$$P = H + 2[p\, \mathrm{e}^J/(\mathrm{e}^J + \mathrm{e}^{-J})]H + 2[p\, \mathrm{e}^J/(\mathrm{e}^J + \mathrm{e}^{-J})]^2 H + \ldots. \tag{4.28}$$

Summing the infinite series gives

$$P = H[(1 + pg)/(1 - pg)] \tag{4.29}$$

where $g = \mathrm{e}^J/(\mathrm{e}^J + \mathrm{e}^{-J})$. Similarly on the rescaled lattice, this probability is

$$P = H' + 2[p'\, \mathrm{e}^{J'}/(\mathrm{e}^{J'} + \mathrm{e}^{-J'})]H' + 2[P\, \mathrm{e}^{J'}/(\mathrm{e}^{J'} + \mathrm{e}^{-J'})]H' + \ldots. \tag{4.30}$$

Again summing the infinite series gives

$$P = H'[(1 + p'g')/(1 - p'g')] \tag{4.31}$$

where $g' = e^{J'}/(e^{J'}+e^{-J'})$. Equating probabilities to preserve the free energy gives

$$H[(1+pg)/(1-pg)] = H'[(1+p'g')/(1-p'g')] \tag{4.32}$$

from which we can determine $f(p, J, b, 1)$ to be

$$f(p, J, b, 1) = [(1+pg)/(1-pg)][(1+p'g')/(1-p'g')]^{-1}. \tag{4.33}$$

In one dimension where all bonds on the lattice are in series, the rescaling of pg is the same in form as (4.2), i.e. $p'g' = p^b g^b$. Using this (4.33) becomes

$$f(p, J, b, 1) = [(1+pg)/(1-pg)]\{[1+(pg)^b]/[1-(pg)^b]\}^{-1} \tag{4.34}$$

which when evaluated at the fixed point (4.10) and (4.12) gives to leading order in ε

$$f(p, J, b, 1) \simeq b[1+(b^{3/2}-b^{1/2})\, e^{-1/\varepsilon}]. \tag{4.35}$$

Using (4.35) and $f(p=1, J=\infty, b, d) = b^d$ gives for the RG equation for H

$$H'/H = b[1+(b^{3/2}-b^{1/2})\, e^{-1/\varepsilon}] + b\varepsilon \ln b. \tag{4.36}$$

This in turn gives for the magnetic eigenvalue y_1^H to leading order in ε

$$y_1^H \simeq 1+\varepsilon+b^{1/2}(b-1)\, e^{-1/\varepsilon}/\ln b. \tag{4.37}$$

Finally from (4.37) we find for η_1

$$\eta_1 \simeq 1-\varepsilon-[2b^{1/2}(b-1)/\ln b]\, e^{-1/\varepsilon} \tag{4.38}$$

or

$$\eta_1 \simeq 2-d-[b^{1/2}(b-1)/\ln b^{1/2}]\, e^{-1/\varepsilon} \tag{4.39}$$

from which we see that η_1 is not zero near $d=1$.

Decimation of the Potts model in an external field forces the introduction of an additional coupling into the Potts Hamiltonian. This coupling is necessary in order to close the RG equations (Stephen 1976, Wu 1977). This method of calculating η_1 avoids that necessity, at least to first order in $\varepsilon = d-1$.

5. Conclusions

A novel approach has been used to obtain a Lagrangian for site–bond percolation. By writing the pair correlation function in terms of the operators $a(\boldsymbol{r})$ and $\bar{a}(\boldsymbol{r})$, eliminating all diagrams with closed loops of arrows, and using a Gaussian transformation, a field theory for the pair correlation was developed.

Using the renormalisation group on this Lagrangian the exponents ν and η were obtained for both percolation and Ising fields. To first order in the $\varepsilon = (6-d)$ expansion the scaling relation $\nu_1 = 2/(d-2)$ was shown to hold. Then, using the equations of motion, this scaling relation was shown to hold to all orders in perturbation theory for $d > 4$.

Within a one-loop calculation, the quartic couplings other than the pure Ising coupling w were shown to be highly irrelevant for $d \leqslant 4$. On this basis and again using the equations of motion, the scaling relation $\nu_1 = 2/(d-2+\eta_2)$ was shown to hold to all orders for $d < 4$.

The exponent ν_1 was calculated to leading order in $\varepsilon = d-1$ using a Migdal–Kadanoff transformation. A new method for calculating the magnetic eigenvalue,

which appeals to the analyticity of the RG equation of the magnetic field for its justification, was used to calculate η, also to leading order in ε.

Although the scaling relation $\nu_1 = 2/(d-2+\eta_2)$ was shown not to hold near $d=1$ and 2, it is still reasonable to assume that it is valid near $d=3$, based on the irrelevance of the quartic couplings for $d \lesssim 4$ and the work of Fucito and Parisi for $d=2$. Since η_2 is small for the 'physical' dimension $d=3$ ($\eta_2 \simeq 0.03$), it is probably safe to assume that $\nu_1 \simeq 2$ for $d=3$.

Acknowledgments

After this work was completed, we heard that A Weinrib has derived a similar scaling relation, as a special case of a more general relation for ν when long-range correlated randomness is present (Weinrib and Halperin 1983). JC thanks A Weinrib for discussions on this point.

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Appendix 1

A1.1.

To carry out the trace over $a(r)$ and $\bar{a}(r)$ we break the integrand of (2.5), i.e. G, into three pieces: those not containing either r_1 or r_2, those containing only r_1, and those containing only r_2. The pair correlation function becomes

$$G(\boldsymbol{r}_1, \boldsymbol{r}_2) = \mathbb{P} \int \mathscr{D}\bar{\varphi}\, \mathscr{D}\varphi \exp\Big(-\sum_{i,j} \bar{\varphi}_i \tilde{V}_{ij}^{-1} \varphi_j\Big) \prod_{r_i \neq r_1 r_2} \operatorname{Tr} \exp(\bar{\varphi}_i a_i + \varphi_i \bar{a}_i)$$
$$\times \operatorname{Tr} a(\boldsymbol{r}_1) \exp(\bar{\varphi}_1 a_1 + \varphi_1 \bar{a}_1) \operatorname{Tr} \bar{a}(\boldsymbol{r}_2) \exp(\bar{\varphi}_2 a_2 + \varphi_2 \bar{a}_2). \tag{A1}$$

Define $F_i = F(\varphi_i, \bar{\varphi}_i)$ by

$$\exp(F(\varphi_i, \bar{\varphi}_i)) \equiv \operatorname{Tr} \exp(\bar{\varphi}_i a_i + \varphi_i \bar{a}_i). \tag{A2}$$

Multiply (A1) by 'one' in the form $1 = \mathrm{e}^{F_1} \mathrm{e}^{-F_1} \mathrm{e}^{F_2} \mathrm{e}^{-F_2}$ to get

$$G(\boldsymbol{r}_1, \boldsymbol{r}_2) = \mathbb{P} \int \mathscr{D}\bar{\varphi}\, \mathscr{D}\varphi \exp\Big(-\sum_{i,j} \bar{\varphi}_i \tilde{V}_{ij}^{-1} \varphi_j\Big) \prod_{\text{all } r_i} \operatorname{Tr} \exp(\bar{\varphi}_i a_i + \varphi_i \bar{a}_i)$$
$$\times \operatorname{Tr} a(\boldsymbol{r}_1) \exp(\bar{\varphi}_1 a_1 + \varphi_1 \bar{a}_1)\, \mathrm{e}^{-F_1} \operatorname{Tr} \bar{a}(\boldsymbol{r}_2) \exp(\bar{\varphi}_2 a_2 + \varphi_2 \bar{a}_2)\, \mathrm{e}^{-F_2}. \tag{A3}$$

Carrying out the trace in (A2) gives

$$\begin{aligned} \exp(F_i) &= \operatorname{Tr}(1 + f(\bar{\varphi}_i) a_i)(1 + f(\varphi_i) \bar{a}_i) \\ &= 1 + n_i f(\bar{\varphi}_i) f(\varphi_i) \\ &= \exp\{\ln[1 + n_i f(\bar{\varphi}_i) f(\varphi_i)]\} \end{aligned} \tag{A4}$$

where

$$f(\varphi_i) = \sum_{n=1}^{\infty} (\mathrm{i})^{m-1} \frac{(\varphi)^m}{m!} \simeq \varphi + \frac{\mathrm{i}\varphi^2}{2!} - \frac{\varphi^3}{3!}. \tag{A5}$$

Similarly, carrying out the trace over the terms in a_1 and a_2 we get, respectively,

$$\begin{aligned}\operatorname{Tr} a_1 \exp(\bar{\varphi}_1 a_1 + \varphi_1 \bar{a}_1)\, \mathrm{e}^{-F_1} &= \operatorname{Tr} a_1(1+f(\bar{\varphi}_1)a_1)(1+f(\varphi_1)\bar{a}_1)\, \mathrm{e}^{-F_1} \\ &= n_1[\, f(\varphi_1)+\mathrm{i}f(\varphi_1)f(\bar{\varphi}_1)]\, \mathrm{e}^{-F_1} \qquad \text{(A6)}\end{aligned}$$

and

$$\begin{aligned}\operatorname{Tr} \bar{a}_2 \exp(\bar{\varphi}_2 a_2 + \varphi_2 \bar{a}_2)\, \mathrm{e}^{-F_2} &= \operatorname{Tr} \bar{a}_2(1+f(\bar{\varphi}_2)a_2)(1+f(\varphi_2)\bar{a}_2)\, \mathrm{e}^{-F_2} \\ &= n_2[\, f(\bar{\varphi}_2)+\mathrm{i}f(\varphi_2)f(\bar{\varphi}_2)]\, \mathrm{e}^{-F_2}. \qquad \text{(A7)}\end{aligned}$$

Finally, substituting the expression for $\tilde{V}^{-1}$, (2.6), the expression for e^{F_1} from (A4), and the two remaining terms contributing to G from (A6) and (A7), we get

$$\begin{aligned}G(\boldsymbol{r}_1, \boldsymbol{r}_2) = \mathbb{P} \int \mathscr{D}\bar{\varphi}\, \mathscr{D}\varphi\, \exp\Big(-\sum_i c_1 \nabla\varphi_i \cdot \nabla\bar{\varphi}_i + c_2\varphi_i\bar{\varphi}_i + \ln[1+nf(\bar{\varphi}_i)f(\varphi_i)]\Big) \\ \times n(r_1)n(r_2)[f(\varphi_1)f(\bar{\varphi}_2)] + \text{higher-order terms} \qquad \text{(A8)}\end{aligned}$$

where c_1, c_2 are constants.

Expanding out the ln term and keeping only the lowest-order terms in $f(\varphi)$ and $f(\bar{\varphi})$, (A8) becomes

$$\begin{aligned}G(\boldsymbol{r}_1, \boldsymbol{r}_2) = \mathbb{P} \int \mathscr{D}\bar{\varphi}\, \mathscr{D}\varphi\, \exp\Big(-\sum_i c_1 \nabla\varphi_i \cdot \nabla\bar{\varphi}_i + c_2\bar{\varphi}_i\varphi_i \\ + n_i[\varphi_i\bar{\varphi}_i - \tfrac{1}{2}\mathrm{i}(\varphi_i\bar{\varphi}_i^2 + \bar{\varphi}_i\varphi_i^2) + \ldots]\Big) n(r_1)\varphi(r_1)n(r_2)\bar{\varphi}(r_2). \qquad \text{(A9)}\end{aligned}$$

A1.2.

Explicitly carrying out the trace over s in (2.14) gives a factor of

$$\begin{aligned}\operatorname{Tr}_s \prod_i \exp\{s_i[\tfrac{1}{2}(\varphi_i\bar{\varphi}_i - \tfrac{1}{2}\mathrm{i}(\varphi_i\bar{\varphi}_i^2 + \bar{\varphi}_i\varphi_i^2)) + \chi_i]\} \\ = \prod_i \exp\{\ln\cosh[\tfrac{1}{2}(\varphi_i\bar{\varphi}_i - \tfrac{1}{2}\mathrm{i}(\varphi_i\varphi_i^2 + \bar{\varphi}_i\varphi_i^2)) + \chi_i]\} \qquad \text{(A10)}\end{aligned}$$

in $\langle G\rangle$. We have ignored the terms with leading factors of s_1, s_2 and $s_1 s_2$ as these lead to higher-order correlation functions in φ, $\bar{\varphi}$ and χ. Expanding the argument of ln as $\cosh\chi \sim 1+\frac{1}{2}\chi^2$, and therefore $\ln\cosh\chi \sim \chi^2/2$, the right-hand side of (A10) becomes

$$\prod_i \exp[\chi_i\varphi_i\bar{\varphi}_i - \tfrac{1}{2}\mathrm{i}\chi_i(\varphi_i\bar{\varphi}_i^2 + \bar{\varphi}_i\varphi_i^2) + \chi_i^2 + \tfrac{1}{4}\varphi_i^2\bar{\varphi}_i^2].$$

This is then substituted into $\langle G\rangle$ and the continuum limit taken to get (2.15).

Appendix 2

To determine if the quartic couplings h, k, n and m are irrelevant, we must consider the β functions of these couplings. If the matrix of coefficients of these β functions is positive definite, then the couplings are irrelevant.

Evaluating the Feynman diagrams for the bare vertex functions to order one loop, we find

$$n_{\mathrm{R}} = Z_\varphi^2 (4\pi)^{-3} \ln(\Lambda/\varkappa)(n - 8g_0^2 n + 6u_0^2 n - 8g_0^2 m - 20u_0 g_0 h + 4u_0^2 k), \qquad \text{(A11)}$$

$$m_R = Z_\varphi^2 (4\pi)^{-3} \ln(\Lambda/\varkappa)(m - 9g_0^2 m + 6u_0^2 m - 3g_0^2 n - 16u_0 g_0 h + 2u_0^2 k), \tag{A12}$$

$$h_R = Z_\chi^{1/2} Z_\varphi^{3/2} (4\pi)^{-3} \ln(\Lambda/\varkappa)(h - 6g_0^2 h + u_0^2 h + g_0 u_0 n + 2g_0 u_0 m + 6g_0 u_0 k), \tag{A13}$$

$$k_R = Z_\chi Z_\varphi (4\pi)^{-3} \ln(\Lambda/\varkappa)(k - g_0^2 k + 5u_0^2 k - 4g_0 u_0 h). \tag{A14}$$

The quartic couplings on the right-hand side of these equations are the bare dimensionful coupling constants.

Evaluating these expressions at the relevant fixed point in g_0 and u_0, (3.12), where the wavefunction renormalisation constants Z_φ and Z_χ are equal to unity, and in terms of the bare dimensionless couplings, (A11)–(A14) become

$$n_R = n_0 \Lambda^{-2+\varepsilon} + \varepsilon \Lambda^{-2+\varepsilon} \ln(\Lambda/\varkappa)(-5n_0 - 8m_0 - 10\sqrt{2}h_0 + 2k_0), \tag{A15}$$

$$m_R = m_0 \Lambda^{-2+\varepsilon} + \varepsilon \Lambda^{-2+\varepsilon} \ln(\Lambda/\varkappa)(-3n_0 - 6m_0 - 8\sqrt{2}h_0 + k_0), \tag{A16}$$

$$h_R = h_0 \Lambda^{-2+\varepsilon} + \varepsilon \Lambda^{-2+\varepsilon} \ln(\Lambda/\varkappa)(\tfrac{1}{2}\sqrt{2}n_0 + \sqrt{2}m_0 - 3h_0 + 3\sqrt{2}k_0), \tag{A17}$$

$$k_R = k_0 \Lambda^{-2+\varepsilon} + \varepsilon \Lambda^{-2+\varepsilon} \ln(\Lambda/\varkappa)(-2\sqrt{2}h_0 + \tfrac{3}{2}k_0). \tag{A18}$$

From these we find for the β functions

$$\beta_n = (2-\varepsilon)n_0 + \varepsilon(5n_0 + 8m_0 + 10\sqrt{2}h_0 - 2k_0), \tag{A19}$$

$$\beta_m = (2-\varepsilon)m_0 + \varepsilon(3n_0 + 6m_0 + 8\sqrt{2}h_0 - k_0), \tag{A20}$$

$$\beta_h = (2-\varepsilon)h_0 + \varepsilon(-\tfrac{1}{2}\sqrt{2}n_0 - \sqrt{2}m_0 + 3h_0 - 3\sqrt{2}k_0), \tag{A21}$$

$$\beta_k = (2-\varepsilon)k_0 + \varepsilon(2\sqrt{2}h_0 - \tfrac{3}{2}k_0), \tag{A22}$$

and thus defining the matrix of coefficients as $\boldsymbol{\beta}'$, we find

$$\boldsymbol{\beta}' = \begin{pmatrix} 2+4\varepsilon & 8\varepsilon & 10\sqrt{2}\varepsilon & -2\varepsilon \\ 3\varepsilon & 2+5\varepsilon & 8\sqrt{2}\varepsilon & -\varepsilon \\ -\tfrac{1}{2}\sqrt{2}\varepsilon & \sqrt{2}\varepsilon & 2+2\varepsilon & -3\sqrt{2}\varepsilon \\ 0 & 0 & 2\sqrt{2}\varepsilon & 2-\tfrac{5}{2}\varepsilon \end{pmatrix}. \tag{A23}$$

It is this matrix which must be checked for positive definiteness. To be positive definite, the four invariants trace, second minor determinant, third minor determinant and the determinant must be greater than zero. For these quantities we find

$$\text{trace} = 8 + 8.5\varepsilon, \tag{A24}$$

$$\text{second minor determinant} = 24 + 51\varepsilon + \tfrac{49}{2}\varepsilon^2, \tag{A25}$$

$$\text{third minor determinant} = 32 + 102\varepsilon + 98\varepsilon^2 - 2\varepsilon^3, \tag{A26}$$

$$\text{determinant} = 16 + 68\varepsilon + 290\varepsilon^2 + 204\varepsilon^3 + 41\varepsilon^4. \tag{A27}$$

As can be readily verified, these are all positive for all $\varepsilon \leqslant 60$. Therefore, if this one-loop calculation can be trusted, the quartic couplings are highly irrelevant for $d \leqslant 4$.

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